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Strategic Information Design in Quadratic Multidimensional Persuasion Games with Two Senders

Raj Kiriti Velicheti, Melih Bastopcu, Tamer Başar

Abstract— **In the classical communication setting, multiple senders having access to the same source of information and transmitting it over channel(s) to a receiver, in general, leads to a decrease in estimation error at the receiver. However, if the objectives of the information providers are different from that of the estimator, this might result in interesting strategic interactions. In this work, we consider a hierarchical signaling game between two senders (information designers) and a single receiver (decision maker) each having their own, possibly misaligned, objectives. The senders lead the game by committing to individual information disclosure policies simultaneously, within the framework of a Nash game among themselves. This is followed by the receiver's action decision. With Gaussian information structure and quadratic objectives (which depend on underlying state and receiver's action) for all the players, we show that in general the equilibrium is not unique. While we show that full revelation of the state is always an equilibrium, we propose an algorithm to achieve non trivial equilibria. Through simulations we show that misalignment between senders' objectives is beneficial for the receiver.**

I. Introduction

Data plays an important role in most modern learning and control systems. Traditionally, data collection and decision making processes are coupled in the same system, eliminating the need to transmit data, and thus suppressing strategic interactions. Recently, with the advancement of communication channels and cheaper data sources, information exchange has been playing a crucial role in the analysis of these systems. For example, consider a robot learning to perform a task based on human demonstrations. In order to perform the task efficiently, a large number of human demonstrations need to be collected, but since humans might have their own biases, this causes a possible mismatch in the objectives [1]. Learning the underlying task amidst these mismatched (possibly strategic) objectives is still a problem of active interest [2]. Similar strategic exchange of information with misaligned objectives also arises in Federated Learning, cyber-security, and many other economic and political interactions [3]–[5].

All the above scenarios comprise multiple senders (information designers), who have access to some information of interest, and a decision maker (receiver) whose action depends on this information. Further, senders' individual objectives would possibly depend on the receiver's action. Hence it is natural to model this as a hierarchical strategic communication game between multiple senders and a decision maker (receiver). As a motivating example, consider two news channels that have access to an underlying true state which is possibly random and multidimensional. If the news sources are biased and have their own objectives, they might benefit from strategic crafting of information. Formalizing these interactions from the lens of game theory helps us address the following questions: What is the effect of adding more senders to the communication game? How do sender biases affect the information learned by the receiver? Analyzing these questions might help understand the proliferation of misinformation through strategic information providers.

The authors are affiliated with the Coordinated Science Lab, University of Illinois at Urbana-Champaign, Urbana, IL, USA 61801. Emails: {rkv4,bastopcu,basar1}@illinois.edu.

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A model of strategic communication was first introduced in the seminal work [6] where, different from the traditional communication literature, their work considers a scenario in which the sender has an additional bias term in the objective which makes this problem a game. This base model has attracted a wide range of follow-up works, including extensions to multiple senders [7], [8] and multidimensional settings [9]. A recent line of work pioneered in [10], [11] (termed as Bayesian Persuasion) alternatively considers a hierarchical signaling game where the receiver responds to a strategy committed by the sender. This commitment power adds benefit to the sender, and hence can be used to characterize the optimal utility that a sender can derive in a communication game. The last decade has seen a surge of interest in this model from various disciplines including control theory [9], [12], machine learning [13], and information theory [14], [15].

Our work adds to the growing literature of multi-sender hierarchical communication games. Reference [16] solves for Nash equilibrium among senders and shows that there can be more than one equilibrium, with full revelation by all senders always being an equilibrium. Studies [17] and [18] consider a slightly different game where senders can commit sequentially and prove that sequential persuasion cannot generate a more informative equilibrium than simultaneous commitment. Both of these works consider finite state and action spaces where state is a one-dimensional variable. In contrast, in the present paper we consider a more general infinite multidimensional state and action spaces while restricting our attention to a Gaussian prior and quadratic utilities. In one of the initial works in this direction reference [19] utilizes semi-definite programming (SDP) to solve a single sender persuasion game by computing a lower bound on the sender's cost [1] and prove that for Gaussian information structures, linear policies can be used to achieve this lower bound. Extensions to dynamic games [20], [21] and priors beyond Gaussian [22] have also been studied in the literature. However, none of these works considers games with multiple senders. [12] considers a multi-sender communication game and solve for a simpler symmetric equilibrium with a large number of senders while restricting attention to senders with symmetric objectives and special prior structure although allowing for private state realization for each sender. [23] provides a more detailed review regarding recent works along this line.

Coming to the specifics of this paper, we pose a strategic communication game between two senders and a receiver. We solve for a hierarchical equilibrium where the senders commit to (possibly different) information disclosure strategies[2] simultaneously (and thus, play a Nash game among themselves) followed by the receiver taking a decision (and as a result, play a Stackelberg game between senders and the receiver) [24]. Due to the presence of two senders, each sender faces an equilibrium computation problem in contrast to an optimization problem as in single sender games discussed in [19], [20]. To compute the hierarchical equilibrium, we pose the problem in the space of posterior covariances, propose a notion of stable posterior and design an algorithm to identify such posteriors. We show that the equilibrium might not be unique and identify a necessary and sufficient condition for a posterior covariance to be in equilibrium. Finally, we provide extensive numerical results to demonstrate the effect of competition in information revelation. In line with [16], a key takeaway from our work is that in games with multidimensional infinite state spaces and quadratic cost[3] functions, the receiver benefits from competition among multiple senders as well.

Notations: $Tr(.)$ denotes the trace of a matrix. We denote vectors with bold lower-case letters. For a given matrix A and a vector $\boldsymbol{y}$, $A^\top$ and $\boldsymbol{y}^\top$ denote the transposes of that matrix and the vector, respectively. The identity and zero matrices are denoted by I and O, respectively. For positive semi-definite matrices A and B, $A \succeq B$ means that $A - B$ is a positive semi-definite matrix. $E_{\ell \times m}$ denotes a matrix of dimensions $\ell \times m$.

II. System Model and Problem Formulation

We consider a non-cooperative game among 3 players, two of them being senders, labeled as 1 and 2, who have access to a state vector $\boldsymbol{x}$, and the third one is a receiver r, as depicted in Fig. 1. The state $\boldsymbol{x} \in \mathbb{R}^p$ is a random variable sampled from a zero mean Gaussian distribution with positive-definite covariance Σ_x, i.e., $\boldsymbol{x} \sim \mathbb{N}(0, \Sigma_x)$ with $\Sigma_x \succ O$. While all objectives and Gaussian prior statistics are common knowledge among the players, the

[1]In line with the control literature, most of the works discussed here consider the scenario where each player wants to minimize their individual cost function rather than maximizing their own utility function.

[2]In this paper, we use the terms strategy and policy interchangeably.

[3]In this paper, we use the terms objective and cost interchangeably.

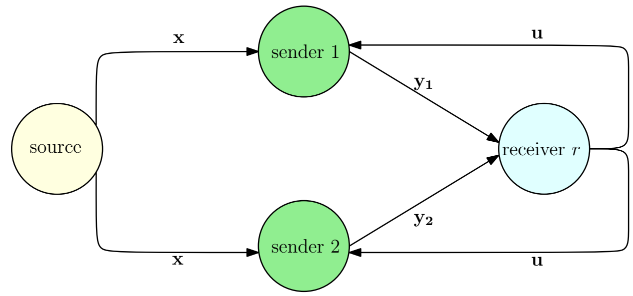


Fig. 1. The strategic communication system consisting of senders 1 and 2, and receiver r.

senders additionally have access to realization of the state $\boldsymbol{x}$, and hence can *design* or shape its transmission to r so as to influence r's choice of action. The game proceeds as follows: At the beginning of the game, the senders ($i = 1, 2$) commit to their individual signaling policies $\eta_i(\cdot) \in \Omega$ simultaneously such that $\boldsymbol{y_i} = \eta_i(\boldsymbol{x})$ where $\boldsymbol{y_i} \in \mathbb{R}^p$ is the message signal and Ω is the policy space which we consider to be the class of all Borel measurable functions from $\mathbb{R}^p$ to $\mathbb{R}^p$. The receiver selects an action $\mathbf{u} = \gamma(\eta_1, \eta_2)(\boldsymbol{y}_1, \boldsymbol{y}_2)$ based on a policy $\gamma \in \Gamma$ where Γ is the set of all Borel measurable functions from $\mathbb{R}^{2p}$ to $\mathbb{R}^t$.

Due to this commitment structure, receiver r's strategy γ can depend on senders' signaling policies, i.e., $\gamma(\boldsymbol{y}_1, \boldsymbol{y}_2) = \gamma(\eta_1, \eta_2)(\boldsymbol{y}_1, \boldsymbol{y}_2)$. Let J_i denote the cost of player i for $i = 1, 2, r$. The hierarchical commitment structure leads to a Nash game played between the senders followed by a Stackelberg game between the senders and the receiver [24]. Let $\Gamma^*(\eta_1, \eta_2)$ denote receiver r's best response set for a given pair of senders' policies (η_1, η_2) which is a subset of Γ, and is given by

$$\Gamma^*(\eta_1, \eta_2) = \operatorname*{argmin}_{\gamma \in \Gamma} J_r(\eta_1, \eta_2, \gamma(\eta_1, \eta_2)).$$

Under certain assumptions such as convexity of the receiver's utility, which is true in our setting as it will be elaborated on in the following sections, receiver's best response set Γ^* forms an equivalence class [24], and hence results in the same random variable $\boldsymbol{u}$ for any $\gamma^* \in \Gamma^*$ almost surely. Therefore, the set of policies $(\eta_1^*, \eta_2^*, \gamma^*)$ are said to achieve equilibrium provided that

$$\begin{aligned} &J_i(\eta_i^*, \eta_{-i}^*, \gamma^*(\eta_i^*, \eta_{-i}^*)) \leq J_i(\eta_i, \eta_{-i}^*, \gamma^*(\eta_i, \eta_{-i}^*)), \\ &\gamma^*(\eta_1, \eta_2) = \operatorname*{argmin}_{\gamma \in \Gamma} J_r(\eta_1, \eta_2, \gamma(\eta_1, \eta_2)), \end{aligned} \tag{1}$$

for $i \in \{1, 2\}$, for all $\eta_i \in \Omega$ where η_{-i}^* denotes the equilibrium strategy of the other sender. Note that in the equilibrium formulation provided in (1), all senders select their strategies independent of the signal realizations. Thus all senders are assumed to minimize their expected costs.

All players have their own individual quadratic (expected) cost functions (objectives) denoted by J_i for $i \in \{1, 2, r\}$, given by

$$J_i(\eta_1, \eta_2, \gamma) = \mathbb{E}[||Q_i\boldsymbol{x} + R_i\boldsymbol{u}||^2], \tag{2}$$

where $i = 1, 2, r$, $Q_i \in \mathbb{R}^{m \times p}$, and $R_i \in \mathbb{R}^{m \times t}$ such that $t \leq m$ and $R_r^\top R_r$ is invertible. By using first order optimality conditions the receiver's (essentially unique) best response $u^* = \gamma^*(\boldsymbol{y_1}, \boldsymbol{y_2}) = -(R_r^\top R_r)^{-1} R_r^\top Q_r \hat{\boldsymbol{x}}(\eta_1, \eta_2)$ where $\hat{\boldsymbol{x}}(\eta_1, \eta_2) = \mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_1, \boldsymbol{y}_2]$ is the posterior estimate of the state. Incorporating this into the senders' objective functions, lets us compute the best response correspondence maps [24]. Particularly, the optimal response of sender i for $i = 1, 2$ to a fixed strategy η_{-i} of the other sender can be obtained by solving the following optimization problem

$$\min_{\eta_i \in \Omega} \mathbb{E}[||Q_i \boldsymbol{x} - R_i (R_r^\top R_r)^{-1} R_r^\top Q_r \hat{\boldsymbol{x}}(\eta_i, \eta_{-i})||^2]. \quad (3)$$

The objective function in (3) is quadratic in $\boldsymbol{x}$, and $\hat{\boldsymbol{x}}$ and senders' policies can only affect the cost through influencing $\hat{\boldsymbol{x}}$. Thus, we can simplify the senders' optimization problem in (3) as

$$\begin{aligned} &\mathbb{E}[||Q_i \boldsymbol{x} - R_i (R_r^\top R_r)^{-1} R_r' Q_r \hat{\boldsymbol{x}}||^2] \\ &= \mathbb{E}[\boldsymbol{x}^\top Q_i^\top Q_i \boldsymbol{x}] - 2\mathbb{E}[\hat{\boldsymbol{x}}^\top \Lambda_i^\top Q_i \boldsymbol{x}] + \mathbb{E}[\hat{\boldsymbol{x}}^\top \Lambda_i^\top \Lambda_i \hat{\boldsymbol{x}}], \end{aligned} \quad (4)$$

where $\Lambda_i = R_i (R_r^\top R_r)^{-1} R_r^\top Q_r$. The first term in (4) does not depend on senders' strategies. By using the law of iterated expectations, the second term in (4) can be rewritten as $\mathbb{E}[\hat{\boldsymbol{x}}^\top \Lambda_i^\top Q_i \boldsymbol{x}] = \mathbb{E}[\mathbb{E}[\hat{\boldsymbol{x}}^\top \Lambda_i^\top Q_i \boldsymbol{x} | \boldsymbol{y}_1, \boldsymbol{y}_2]] = \mathbb{E}[\hat{\boldsymbol{x}}^\top \Lambda_i^\top Q_i \mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_1, \boldsymbol{y}_2]] = \mathbb{E}[\hat{\boldsymbol{x}}^\top \Lambda_i^\top Q_i \hat{\boldsymbol{x}}]$. Then, we can rewrite the optimization problem in (3) equivalently as $\min_{\eta_i \in \Omega} \mathbb{E}[\hat{\boldsymbol{x}}^\top V_i \hat{\boldsymbol{x}}]$, where

$$V_i = \Lambda_i^\top \Lambda_i - \Lambda_i^\top Q_i - Q_i^\top \Lambda_i. \quad (5)$$

Without loss of generality, we take the first moment of $\hat{\boldsymbol{x}}$ to be equal to $\mathbb{E}[\boldsymbol{x}] = 0$. The posterior covariance denoted by Σ is given by $\mathbb{E}[||\hat{\boldsymbol{x}} - \mathbb{E}[\hat{\boldsymbol{x}}]||^2] = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top]$.

To illustrate the above simplification, we provide an example below where the state of the world $\boldsymbol{x} \in \mathbb{R}^3$ consists of 3 scalar random variables z, θ_A, and θ_B.

Example 1 *Let us re-visit the example of the information disclosure game between two news providers. To accommodate biases in the news providers, suppose that each news provider wants to deceive the receiver about the true state. We consider the state of the world as $\boldsymbol{x} = \begin{bmatrix} z & \theta_A & \theta_B \end{bmatrix}^\top$, where z represents the true state which the receiver is interested in, and θ_A, θ_B denote the innate biases that each news provider has. Formally, let sender 1 have a cost given by $J_1 = \mathbb{E}[|z + \theta_A - u|^2]$ and the sender 2 have a cost given by $J_2 = \mathbb{E}[|z + \theta_B - u|^2]$. On the other hand, the receiver is only interested in z, and thus has the objective function $J_r = \mathbb{E}[|z - u|^2]$. With these objective functions, we have the following matrices: $Q_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $R_1 = -1$, $Q_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, $R_2 = -1$, $Q_r = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $R_r = -1$.*

Thus, by using the definition of V_i in (5), we have

$$V_1 = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Later, in Section IV, we provide a complete solution to Example 1.

III. Equilibrium Computation

While (3) offers a good starting point to compute the best response for each sender, in its present form this is a functional optimization problem for each sender on Borel measurable functions in Ω which can be intractable. Thus, in order to find these functions, in this section, we first reformulate the optimization problem of each sender in (3) to a semi-definite program (SDP). Then, we provide an optimal solution to the SDP formulation which we then prove can be achieved using linear plus noise policies, and thus provide the optimal solution to the original problem in (3). In what follows, we first state and prove a few lemmas which will help identify the reaction set of sender i for a fixed strategy of the other sender $-i$.

Lemma 1 *For a fixed strategy of the other sender $-i$, if the posterior covariance at the receiver is Σ_{-i} (i.e., $\mathbb{E}[\boldsymbol{x}|\boldsymbol{y_{-i}}]\mathbb{E}[\boldsymbol{x}|\boldsymbol{y_{-i}}]^\top = \Sigma_{-i}$), sender i can only induce posterior covariance Σ in the set $\mathcal{S}(\Sigma_{-i}) := \{\Sigma \mid \Sigma_x \succeq \Sigma \succeq \Sigma_{-i}\}$, holding for $i = 1, 2$.*

Proof: Each sender's strategy η_i induces a posterior covariance matrix which by definition is positive semi-definite. Thus, it follows that $\mathbb{E}[(\boldsymbol{x} - \hat{\boldsymbol{x}})(\boldsymbol{x} - \hat{\boldsymbol{x}})^\top] = \Sigma_x - \Sigma \succeq O$. Further,

$$\begin{aligned} &\mathbb{E}[(\hat{\boldsymbol{x}} - \mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}])(\hat{\boldsymbol{x}} - \mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}])^\top] = \mathbb{E}[\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top] \\ &\quad - 2\mathbb{E}[\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}]\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}, \boldsymbol{y}_i]^\top] + \mathbb{E}[\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}]\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}]^\top]. \end{aligned}$$

Using the law of iterated expectations, the second term is

$$\mathbb{E}[\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}]\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}, \boldsymbol{y}_i]^\top] = \mathbb{E}[\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}]\mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}]^\top].$$

Utilizing this, we have $\mathbb{E}[(\hat{\boldsymbol{x}} - \mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}])(\hat{\boldsymbol{x}} - \mathbb{E}[\boldsymbol{x}|\boldsymbol{y}_{-i}])^\top] = \Sigma - \Sigma_{-i} \succeq O$. Hence, we obtain $\Sigma_x \succeq \Sigma \succeq \Sigma_{-i}$. ■

Therefore the posterior covariance matrix Σ induced by sender i's policy η_i should at least follow the condition provided in Lemma 1. This result in Lemma 1 is intuitive. If sender i's policy is to reveal the state of the world $\boldsymbol{x}$ entirely, then the posterior covariance induced by this strategy is $\Sigma = \Sigma_x$. On the other hand, if sender i chooses a policy with no information disclosure, then the receiver can always use the signal from sender $-i$ and as a result, its posterior covariance is equal to $\Sigma = \Sigma_{-i}$. Utilizing this, in the following lemma we reformulate the optimization problem in (3) as a semi-definite program.

Lemma 2 *An equivalent SDP formulation of the optimization problem in (3) is given by*

$$\begin{aligned} \min_{S \in \mathbb{S}^p} \quad & Tr(V_i S) + Tr(Q_i^\top Q_i \Sigma_x) \\ \text{s.t.} \quad & \Sigma_x \succeq S \succeq \Sigma_{-i} \succeq O, \end{aligned} \quad (6)$$

where $\mathbb{S}^p$ denotes the space of positive semi-definite matrices with dimension $p \times p$. Moreover, the optimization problem in (6) is a convex optimization problem.

Proof: Since $\mathbb{E}[\hat{\boldsymbol{x}}^\top V_i \hat{\boldsymbol{x}}] = Tr(\mathbb{E}[V_i \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top])$, the optimization problem in (3) for sender i for $i = 1, 2$, can be written as $\min_{\eta_i \in \Omega} Tr(\mathbb{E}[V_i \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top]) + Tr(Q_i^\top Q_i \Sigma_x) =$

$\min_{\eta_i \in \Omega} Tr(V_i\Sigma) + Tr(Q_i^\top Q_i \Sigma_x)$. Therefore the senders' strategies affect the cost only through the second moment of the posterior estimate.

Note that for a given strategy of sender $-i$, the constraint $\Sigma_x \succeq S \succeq \Sigma_{-i}$ is a necessary condition for S to satisfy because of Lemma 1. The fact that this is sufficient follows from techniques similar to those in [19], [22]. More precisely, given any S satisfying $\Sigma_x \succeq S \succeq \Sigma_{-i}$, let $S = U_x^\top \Lambda_x^{\frac{1}{2}} T \Lambda_x^{\frac{1}{2}} U_x$ where $\Sigma_x = U_x \Lambda_x U_x^\top$, which implies $I \succeq T \succeq O$. Further, let $T = U_t^\top \Lambda_t U_t$, hence $\Lambda_{t,i} \in [0,1]$ where $\Lambda_t = diag(\Lambda_{t,1}, \ldots, \Lambda_{t,p})$.

Considering a linear plus noise policy given by $\eta(\boldsymbol{x}) = L\boldsymbol{x} + w$ where $w \sim \mathbb{N}(0, W)$, with $L = \Lambda_\ell^\top U_t^\top \Lambda_x^{-\frac{1}{2}} U_x^\top$ and $W = diag(w_1^2, \ldots, w_p^2)$ with $\frac{(\Lambda_{\ell,i})^2}{(\Lambda_{\ell,i})^2 + w_i^2} = \Lambda_{t,i}$ completes the proof. Thus, the SDP problem (6) is equivalent to the original optimization problem of each sender in (3).

To prove convexity of the constraint, consider $M_1, M_2 \in \mathbb{S}^p, M_1 \neq M_2$ such that $\Sigma_x \succeq M_1 \succeq \Sigma_{-i}$ and $\Sigma_x \succeq M_2 \succeq \Sigma_{-i}$. Since for any $\alpha \in [0,1]$, the linear combination $E(\alpha) = \alpha M_1 + (1-\alpha) M_2$ satisfies $\Sigma_x \succeq E(\alpha) \succeq \Sigma_{-i}$, we conclude that the set $\Sigma_x \succeq S \succeq \Sigma_{-i}$ in (6) is a convex set. As $Tr(.)$ operator is linear in S, we conclude that (6) is a convex optimization problem. ■

The proof of Lemma 2 provides explicit construction of linear plus noise policies for the senders which can achieve any posterior $S \in \mathbb{S}^p$ such that $\Sigma_x \succeq S \succeq O$. Next, we introduce a few quantities that build on the above two lemmas and are useful to identify the equilibrium posteriors.

Definition 1 *A posterior covariance Σ is said to be stable for sender i if*

$$\min_{\Sigma_x \succeq S \succeq \Sigma \succeq O} Tr(V_i S) = Tr(V_i \Sigma).$$

For the optimization problem in (6), at a stable equilibrium Σ, none of the senders want to reveal any more information.

Lemma 3 *If a posterior covariance Σ^* is stable and achievable by both the senders, then Σ^* is an equilibrium posterior covariance.*

Proof: Consider a posterior covariance Σ^* which is stable for both the senders. From proof of Lemma 2, there exists a strategy η_i' for each sender $i = 1, 2$ which can induce this posterior covariance Σ^*. Consider the strategy profile (η_1', η_2') such that $\mathbb{E}[\hat{x}\hat{x}^\top | y_1, y_2] = \mathbb{E}[\hat{x}\hat{x}^\top | y_1] = \mathbb{E}[\hat{x}\hat{x}^\top | y_2]$. If Σ^* is a stable posterior covariance for sender i, from Definition 1, we have

$$\min_{\Sigma_x \succeq S \succeq \Sigma^*} Tr(V_i S) = Tr(V_i \Sigma^*).$$

Since η_{-i}' induces Σ^*, which is stable for sender i, η_i' belongs to sender i's best response set (because it induces the same posterior Σ^*). Hence (η_1', η_2') is an equilibrium strategy profile resulting in the posterior covariance Σ^*. ■

Thus, to identify the Nash equilibrium among senders for the strategic communication game, it is sufficient to identify posteriors which are stable and achievable by all senders. To identify such posteriors, we reformulate the SDP problem given in (6) using the following lemma.

Lemma 4 *For any Σ' such that $\Sigma_x \succeq \Sigma' \succeq O$, the optimization problem*

$$\begin{aligned} \min_{S \in \mathbb{S}^p} \quad & Tr(V_i S) \\ \textit{s.t.} \quad & \Sigma_x \succeq S \succeq \Sigma', \end{aligned} \tag{7}$$

can be equivalently written as

$$\begin{aligned} \min_{Z \in \mathbb{S}^p} \quad & Tr(W_i Z) + Tr(V_i \Sigma') \\ \textit{s.t.} \quad & I \succeq Z \succeq O, \end{aligned} \tag{8}$$

where $W_i = (\Sigma_x - \Sigma')^{\frac{1}{2}} V_i (\Sigma_x - \Sigma')^{\frac{1}{2}}$.

Proof: Let $Z := I - K^2$, with $K = \left((\Sigma_x - \Sigma')^{\frac{1}{2}}\right)^\dagger (\Sigma_x - S)^{\frac{1}{2}}$, where $(.)^\dagger$ denotes the pseudo-inverse of a matrix.[4] Further, let $\lambda(K)$ denote the largest eigenvalue of K. Then $\lambda(K) \in [0,1]$ [25]. By using the fact that any non-zero vector is an eigenvector of identity matrix, it follows that $\lambda(Z) \in [0,1]$. Thus, we have $I \succeq Z \succeq O$.

Next, for ease of exposition, we denote $\Sigma_x - S$ by Q, i.e., $Q = \Sigma_x - S$. Then, we have $(\Sigma_x - \Sigma')^{\frac{1}{2}} K = Q^{\frac{1}{2}}$ and Q can be equivalently written as

$$Q = (\Sigma_x - \Sigma')^{\frac{1}{2}} K^2 (\Sigma_x - \Sigma')^{\frac{1}{2}}. \tag{9}$$

Since $Z = I - K^2$, Q in (9) is equal to

$$Q = (\Sigma_x - \Sigma') - (\Sigma_x - \Sigma')^{\frac{1}{2}} Z (\Sigma_x - \Sigma')^{\frac{1}{2}}. \tag{10}$$

Then, we have $S = \Sigma' + (\Sigma_x - \Sigma')^{\frac{1}{2}} Z (\Sigma_x - \Sigma')^{\frac{1}{2}}$. Thus, the objective function $Tr(V_i S)$ in (7) is equal to the objective function $Tr(W_i Z) + Tr(V_i \Sigma')$ in (8) where $W_i = (\Sigma_x - \Sigma')^{\frac{1}{2}} V_i (\Sigma_x - \Sigma')^{\frac{1}{2}}$ which completes the proof. ■

Based on Lemmas 3 and 4, we have the following result:

Proposition 1 *If $Tr(V_i \Sigma_i^*) = \min_{\Sigma_x \succeq S \succeq O} Tr(V_i S)$, then $W_i^* = (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} V_i (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} \succeq O$.*

Proof: We note that if $Tr(V_i \Sigma_i^*) = \min_{\Sigma_x \succeq S \succeq O} Tr(V_i S)$, then we have

$$Tr(V_i \Sigma_i^*) = \min_{\Sigma_x \succeq S \succeq \Sigma_i^*} Tr(V_i S). \tag{11}$$

Then, by using Lemma 4, we rewrite (11) as

$$\begin{aligned} \min_{Z \in \mathbb{S}^p} \quad & Tr(W_i^* Z) + Tr(V_i \Sigma_i^*) \\ \text{s.t.} \quad & I \succeq Z \succeq O, \end{aligned} \tag{12}$$

where $W_i^* = (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} V_i (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}}$. Since the minimum value in (12) is equal to $Tr(V_i \Sigma_i^*)$, we must have $\min_{I \succeq Z \succeq O} Tr(W_i Z) = 0$ which happens when the symmetric matrix W_i^* is positive semi-definite, i.e., $W_i^* = (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} V_i (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} \succeq O$. To show this, suppose that there exists a $Q \in \mathbb{R}^{p \times r}$ where $r > 0$ such that $Z = QQ^\top$.

[4] We note that since $\Sigma_x - \Sigma'$ might not be invertible, we use pseudo-inverse of a square matrix, which always exists.

Then, we get $Tr(W_i^* Z) = Tr(Q^\top W_i^* Q) = \sum_{\ell=1}^r q_\ell^\top W_i^* q_\ell$, where q_ℓ is the ℓth column of Q. We note that if W_i^* is not a positive semi-definite matrix, then we can find a Q matrix such that $Tr(W_i^* Z)$ is less than 0. However, since we have $\min_{I \succeq Z \succeq O} Tr(W_i^* Z) = 0$, we conclude that W_i^* is a positive semi-definite matrix. ■

In Proposition 1, we have shown that when there is no constraint on the posterior covariance in the optimization problem given by (6), i.e., when $\Sigma_{-i} = O$, the overall minimum value achieved in (6) is given by $Tr(V_i \Sigma_i^*)$. Further, such a Σ_i^* in Proposition 1 may not be unique. In other words, there can be multiple Σ_i^*s with $W_i^* \succeq O$ and achieve the same minimum value $Tr(V_i \Sigma_i^*)$. Moreover, such Σ_i^* may not be orderable. We utilize tools developed thus far to prove that any $\Sigma' \succeq \Sigma_i^*$ is stable for sender i.

Lemma 5 *Let $Tr(V_i \Sigma_i^*) = \min_{\Sigma_x \succeq S \succeq O} Tr(V_i S)$. Then, for any Σ' such that $\Sigma_x \succeq \Sigma' \succeq \Sigma_i^* \succeq O$, we have*

$$\min_{\Sigma_x \succeq S \succeq \Sigma'} Tr(V_i S) = Tr(V_i \Sigma'). \tag{13}$$

Proof: By using Lemma 4, we have

$$\min_{\Sigma_x \succeq S \succeq \Sigma'} Tr(V_i S) = \min_{I \succeq Z \succeq O} Tr(W_i Z) + Tr(V_i \Sigma'),$$

where $S = \Sigma' + (\Sigma_x - \Sigma')^{\frac{1}{2}} Z (\Sigma_x - \Sigma')^{\frac{1}{2}}$ and $W_i = (\Sigma_x - \Sigma')^{\frac{1}{2}} V_i (\Sigma_x - \Sigma')^{\frac{1}{2}}$. By definition, $\Sigma_x - \Sigma_i^* \succeq \Sigma_x - \Sigma'$. Hence, there exists a K' with $\lambda(K') \subset [0,1]$ such that $(\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} K' = (\Sigma_x - \Sigma')^{\frac{1}{2}} \succeq O$ [25]. Using the unique positive semi-definite square root of a positive semi-definite matrix, we have

$$\begin{aligned} W_i &= (\Sigma_x - \Sigma')^{\frac{1}{2}} V_i (\Sigma_x - \Sigma')^{\frac{1}{2}} \\ &= K' (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} V_i (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} K' \succeq O, \end{aligned}$$

since $W_i^* = (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} V_i (\Sigma_x - \Sigma_i^*)^{\frac{1}{2}} \succeq O$ from Proposition 1. Thus, $\min_{I \succeq Z \succeq O} Tr(W_i Z)$ is equal to 0, and we have $\min_{\Sigma_x \succeq S \succeq \Sigma'} Tr(V_i S) = Tr(V_i \Sigma')$. ■

Lemma 5 hints at the possibility of existence of multiple equilibria. To this end, we identify the set of stable posteriors:

Proposition 2 *The full set of posterior covariances that are stable for sender i for $i = 1, 2$ is given by the set $\{\Sigma' \mid W_i = (\Sigma_x - \Sigma')^{\frac{1}{2}} V_i (\Sigma_x - \Sigma')^{\frac{1}{2}} \succeq O\}$.*

Proof: By using Lemma 4, the optimization problem $\min_{\Sigma_x \succeq S \succeq \Sigma'} Tr(V_i S)$ can be equivalently written as $\min_{I \succeq Z \succeq O} Tr(W_i Z) + Tr(V_i \Sigma')$. If $W_i = (\Sigma_x - \Sigma')^{\frac{1}{2}} V_i (\Sigma_x - \Sigma')^{\frac{1}{2}} \succeq O$, the minimum value is equal to $Tr(V_i \Sigma')$. Thus, every Σ' that generates $W_i \succeq O$ is stable for sender i. ■

In particular, by using Lemma 5 and Proposition 2, we can show that any posterior covariance $\Sigma^* \in \cap_{i=1}^2 \mathcal{D}_i$ where $\mathcal{D}_i = \{\Sigma' \mid \Sigma_x \succeq \Sigma' \succeq \Sigma_i^*\}$ is a stable posterior, which are also achievable using linear plus noise policies as proved in Lemma 2.

Proposition 3 *Full information revelation is always an equilibrium.*

Proof: The proof follows directly from noticing that $W_i = O \succeq O$, for $i = \{1, 2\}$. Since Σ_x is stable and achievable for all senders, it is also an equilibrium posterior covariance. ■

This is in line with the observations of [16] for finite state and action spaces. It can be seen that even if both senders have identical preferences, full revelation is an equilibrium outcome.[5] While Proposition 2 identifies a necessary and sufficient condition for the set of equilibrium posteriors, it only provides a way to verify if a given posterior is an equilibrium posterior but does not provide a constructive approach to identify an equilibrium posterior. In the following sub-section we propose an algorithm to identify an equilibrium posterior explicitly.

Sequential Optimization to Find Equilibrium Posteriors

We now propose an algorithm to identify admissible equilibrium posteriors for the two-sender game. The algorithm uses the standard SDP solver in [26] with minor variations.

Since V_i in general can be singular, there can be multiple equivalent (minimal) solutions to the SDP problem (at line 4) in the Algorithm 1. In order to provide the largest feasible posterior set among these solutions, we choose the one obtained from the following optimization problem

$$\begin{aligned} \min_{\Sigma_x \succeq S_i' \succeq O} \quad & ||\Sigma_x^{-\frac{1}{2}} S_i' \Sigma_x^{-\frac{1}{2}}||_* \\ \text{s.t.} \quad & Tr(V_i S_i) \geq Tr(V_i S_i'), \end{aligned} \tag{14}$$

where $||.||_*$ denotes the nuclear norm which is equal to summation of the singular values of a matrix that is used to obtain a low rank solution in (14). Since the optimization variable in (14) is a symmetric positive semi-definite matrix, the nuclear norm in this case is equal to summation of the eigenvalues of the matrix $\Sigma_x^{-\frac{1}{2}} S_i' \Sigma_x^{-\frac{1}{2}}$ which are in $[0,1]$. To be more precise, the eigenvalues of $\Sigma_x^{-\frac{1}{2}} S_i' \Sigma_x^{-\frac{1}{2}}$ are equal to 1 corresponding to the negative eigenvalues of $W_1 = \Sigma_x^{\frac{1}{2}} V_1 \Sigma_x^{\frac{1}{2}}$ and take arbitrary values in $[0,1]$ for the 0 eigenvalues of W_1. The nuclear norm minimization problem (14) is convex [27] and finds an equilibrium posterior where $\Sigma_x^{-\frac{1}{2}} S_i' \Sigma_x^{-\frac{1}{2}}$ is a projection matrix.

Proposition 4 *The posterior Σ^* obtained from Algorithm 1 is an equilibrium posterior.*

Proof: The fact that Σ^* is stable for sender 1 follows directly from Lemma 5 and $\Sigma^* \succeq \Sigma_1^*$. Further, since Σ^* is obtained from minimization in line (9), (10) of Algorithm 1, using Lemma 4, we directly obtain $(\Sigma_x - \Sigma^*)^{\frac{1}{2}} V_2 (\Sigma_x - \Sigma^*)^{\frac{1}{2}} \succeq O$, and thus from Proposition 2, Σ^* is stable for sender 2. Due to Lemma 2, Σ^* is achievable using linear plus noise policies. Therefore, Σ^* is an equilibrium posterior. ■

Due to the sequential nature of the algorithm, for a two sender problem, we are able to find two admissible equilibrium posteriors. Further, the two equilibrium posteriors

[5]In case of identical objective functions for both senders, such a full revelation policy may not be admissible, in the sense that there can be other policies (different from full revelation) for the senders (one being the optimal single sender policy adopted by both) which would lead to lower average cost for the senders.

Algorithm 1 Finding an Equilibrium Posterior via SDP

Parameters: Σ_x
Compute $V_i \ \forall i \in \{1,2\}$ by using (5)
Variables: $S_i, S_i' \in \mathbb{S}^p$ with $S_i, S_i' \succeq O \ \forall i \in \{1,2\}$
Solve: $\min_{\Sigma_x \succeq S_1 \succeq O} Tr(V_1 S_1)$, by using CVX
Solve: $\min_{\Sigma_x \succeq S_1' \succeq O} ||\Sigma_x^{-\frac{1}{2}} S_1' \Sigma_x^{-\frac{1}{2}}||_*$,
    s.t. $Tr(V_1 S_1) \succeq Tr(V_1 S_1')$ by using CVX
$\Sigma_1^* \leftarrow S_1'$
Solve: $\min_{\Sigma_x \succeq S_2 \succeq \Sigma_1^*} Tr(V_2 S_2)$, by using CVX
Solve: $\min_{\Sigma_x \succeq S_2' \succeq \Sigma_1^*} ||\Sigma_x^{-\frac{1}{2}} S_2' \Sigma_x^{-\frac{1}{2}}||_*$,
    s.t. $Tr(V_2 S_2) \succeq Tr(V_2 S_2')$ by using CVX
Return: $\Sigma^* \leftarrow S_2'$

obtained can yield different costs to the players. Here, we emphasize that the sequential nature of the algorithm is just to find an admissible and stable posterior but this does not imply anything about commitment orders for policy computation. This posterior can be achieved by simultaneous commitment by the senders to the same linear plus policies given by Lemma 2.

IV. Numerical Results

In this section, we provide simulation results to analyze in several examples the effects of changing correlations and alignments among senders in the persuasion game. In these numerical results, we consider a state of the world given by the 3-dimensional vector $\boldsymbol{x} = \begin{bmatrix} z & \theta_A & \theta_B \end{bmatrix}^\top$. For all the examples considered, we take the cost functions of the players given as

$$J_1 = \mathbb{E}[|z + \beta\theta_A + \alpha\theta_B - u|^2], \tag{15}$$
$$J_2 = \mathbb{E}[|z + \alpha\theta_A + \beta\theta_B - u|^2], \tag{16}$$
$$J_r = \mathbb{E}[|z - u|^2], \tag{17}$$

where we specify α and β values for each example. We note that both senders having access to the full state allows us to have such objectives for the sender which depend on both θ_A, θ_B.

Note that as proved in Section III (Proposition 2) there can be multiple equilibria in each of the following situations that would be described below, However, to derive insights from the model, we analyze the costs that result in the equilibrium attained by Algorithm 1.

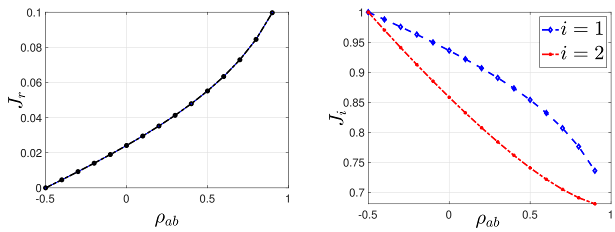
1) Construction of Nash equilibrium policies: We first consider $\alpha = 0$, and $\beta = 1$ and a zero mean Gaussian prior on the state of the world with Σ_x given by

$$\Sigma_x = \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.5 & 1.5 & 0.2 \\ 0.7 & 0.2 & 1 \end{bmatrix}. \tag{18}$$

By using Algorithm 1, we get a stable equilibrium posterior

$$\Sigma^* = \begin{bmatrix} 0.9915 & 0.5171 & 0.7546 \\ 0.5171 & 1.4659 & 0.0909 \\ 0.7546 & 0.0909 & 0.6508 \end{bmatrix}. \tag{19}$$

This achieves a cost of $J_1 = -2.02$, $J_2 = -3.96$. From Lemma 2, we can construct Nash equilibrium policies to be



(a) (b)

Fig. 2. The cost of the (a) receiver, (b) senders with respect to ρ_{ab}.

$\eta_1(\boldsymbol{x}) = \eta_2(\boldsymbol{x}) = L\boldsymbol{x}$, where

$$L = \begin{bmatrix} -0.9093 & 0.0756 & -0.1657 \\ 0.0967 & -0.8367 & 0.2766 \\ 0 & 0 & 0 \end{bmatrix}.$$

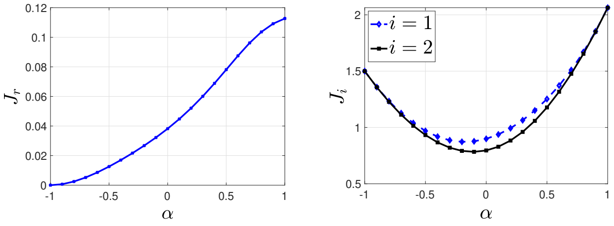
Notice that the senders' strategies turned out to be noiseless ($W = O$). Considering this insight, we aim to further explore in future work whether we can in general restrict our attention to linear policies for equilibrium posteriors obtained using Algorithm 1.

2) Correlation between the state of the world parameters: In the second numerical example, we consider the same setting as in the first example, i.e., $\alpha = 0$, and $\beta = 1$, but with a different covariance matrix Σ_x. Here, we take

$$\Sigma_x = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & \rho_{ab} \\ 0.5 & \rho_{ab} & 1 \end{bmatrix}, \tag{20}$$

where ρ_{ab} is the correlation coefficient between θ_A and θ_B which is given by $\rho_{ab} = \frac{Cov(\theta_A,\theta_B)}{\sqrt{Var(\theta_A)Var(\theta_B)}}$. In this example, we vary $\rho_{ab} = \{-0.5, -0.4, \cdots, 0.9\}$ and find the cost of receiver and the senders obtained by Algorithm 1. In Fig. 2(a), we see how the error covariance of the receiver, i.e., $J_r = \mathbb{E}[||z - u||^2]$, changes with respect to ρ_{ab}. When $\rho_{ab} < 0$, θ_A, and θ_B are negatively correlated, and as a result, senders reveal more information to the receiver about the state of the world $\boldsymbol{z}$. As the correlation ρ_{ab} increases, the information revealed to the receiver decreases and the cost of the receiver J_r increases. In Fig. 2(b), we see how the cost of the receiver changes with respect to ρ_{ab}. As the senders' goals become aligned, they can manipulate the receiver more, and decrease their cost effectively.

3) Correlation between senders' objectives: In the third numerical example, we consider Σ_x in (20) with $\rho_{ab} = 0.25$. In this example, we take $\beta = 1$ and vary $\alpha \in [-1, 1]$. Thus, the senders' objective functions include both the parameters θ_A and θ_B where we have $J_1 = \mathbb{E}[||z+\theta_A+\alpha\theta_B-u||^2]$ and $J_2 = \mathbb{E}[||z+\alpha\theta_A+\theta_B-u||^2]$. In this example, α represents the alignment of interests in θ_A, θ_B for the senders. As in the previous examples, the receiver is only interested in z and thus, it has the objective function $J_r = \mathbb{E}[||z-u||^2]$. We see in Fig. 3(a) that when $\alpha = -1$, i.e., when the senders' interests are exactly mismatched, the receiver can use this to its advantage and recover z perfectly and thus, $J_r = 0$ when $\alpha = -1$. As α increases, the correlation between the senders' objectives increases. As a result, the receiver



(a) (b)

Fig. 3. The cost of the (a) receiver, (b) senders with respect to α.

gets less information about the parameter z, and its cost J_r increases with α. When there is an exact mismatch between the senders' interests, i.e., when $\alpha = -1$, we see in Fig. 3(b) that the senders' costs are high, as the receiver can recover z perfectly. As α gets closer to 0, the senders' objective functions become less dependent, and as a result, senders can achieve their minimum cost. As the senders' objective functions are positively correlated, i.e., when $\alpha > 0$, the cost is increasing for both senders.

V. Conclusion and Discussion

In this paper, we considered a strategic communication game between 2 senders and a receiver with misaligned objective functions. We first posed the senders' optimization problems in terms of the receiver's posterior covariance. Then, by constructing an equivalent SDP, we proposed a sequential optimization technique to find the stable posteriors which can be achieved by using linear plus noise policies, and as a result, they are in fact the equilibrium policies for the senders. Through simulations, we showed that having two senders is beneficial for the receiver as its estimation error cannot get worse as we go from one to two senders. Further, it is beneficial for the receiver to have senders with maximum misalignment between their objectives. As an interesting future direction, we plan to analyze theoretically, the effect of varying alignment between the senders on the set of equilibrium posteriors.

Although this paper deals with a game between 2 senders and a receiver, similar tools can be used to extend this to a general $m > 2$ sender communication game. In particular, a sequential optimization technique, in line with Algorithm 1, can be designed considering objectives of all senders. This can help analyze both theoretically and empirically the effect of adding senders on the set of equilibria. We will extend our results to the most general case with $m > 2$ senders in a future journal version of this work in addition to investigating the optimality of purely linear strategies for the senders. Extension of our work to account for noisy communication channels between the senders and the receiver is also an interesting future direction.

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